A Reproducible Constraint-Pipeline Formulation of Unique Continuation for Schrödinger Equations

Dan Hawkley

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Abstract

We give a reproducible, figure-driven formulation of unique continuation for Schrödinger equations. The standard argument—two-time Gaussian decay, Carleman-weighted norms, Hardy-type variation constraints, and contradiction—is preserved while being organized as a computational pipeline.

Companion notebooks generate all figures and compute finite-grid realizations of weighted norms, gradient costs, and curvature effects, providing direct representations of the analytical structure. These quantities make explicit how non-zero candidates generate incompatible constraint costs, while the zero solution remains the only configuration consistent with all constraints.

The result is an aligned system of analysis, computation, and visualization that clarifies the structure of the classical proof without altering its mathematical content.

1 Introduction

Unique continuation asks when local decay forces global structure. For Schrödinger equations, sufficiently strong decay at two distinct times implies that the solution must be identically zero.

We present a constraint pipeline:

$$\text{two-time decay} \to \text{Carleman weighting} \to \text{Hardy variation cost} \to \text{contradiction} \to \text{zero solution.}$$

Figures are generated from companion notebooks, ensuring alignment between explanation, computation, and visualization.

The goal is clarity: same math, lifted structure.

2 Theorem and assumptions

We consider

$$i\partial_t u = -\Delta u + V(x)u.$$

Theorem 1. *If a solution exhibits sufficiently strong Gaussian decay at two distinct times, then* $u \equiv 0$.

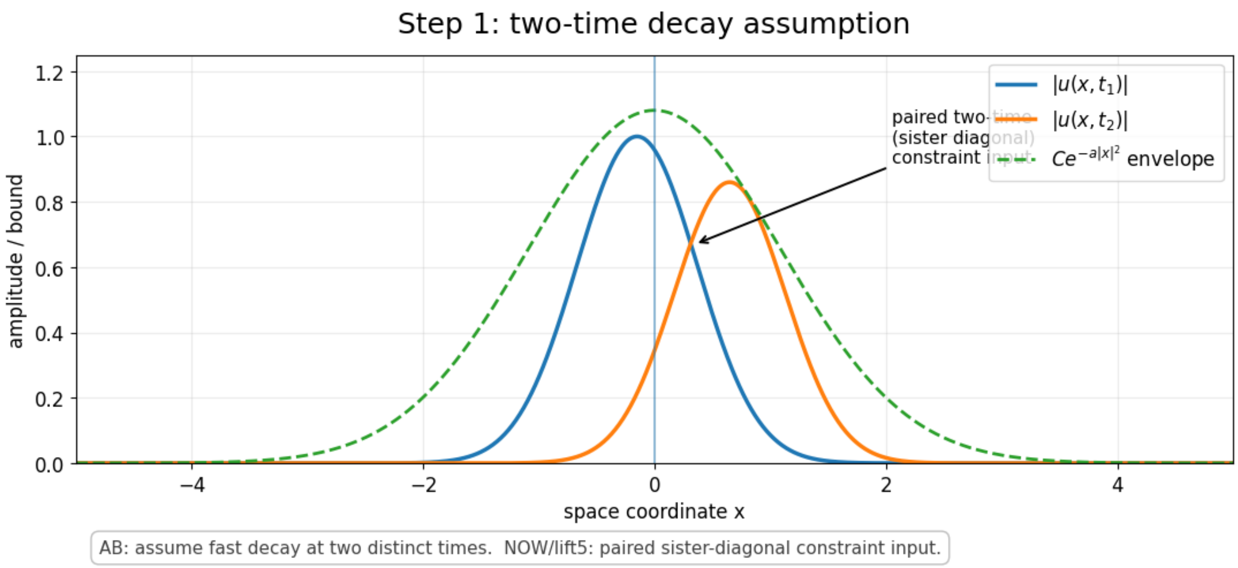

Figure 1: Step 1. Two-time decay: the solution is small at two distinct times, forming a paired constraint input.

3 Carleman weights

Carleman estimates introduce exponential weights:

$$v = e^{\phi}u.$$

These transform decay into global constraints.
Interpretation. Smallness acts under constraints.

4 Hardy-type inequalities

A model inequality:

$$\int \frac{|u|^2}{|x|^2} \leq C \int |\nabla u|^2.$$

Interpretation. Concentration must pay variation cost.

5 Contradiction chain

We combine:

- two-time decay,
- Carleman-weighted norms,
- Hardy variation cost,
- PDE curvature constraints.

Non-zero candidates generate positive weighted, variation, and curvature costs.
Conclusion. Only the zero solution remains.

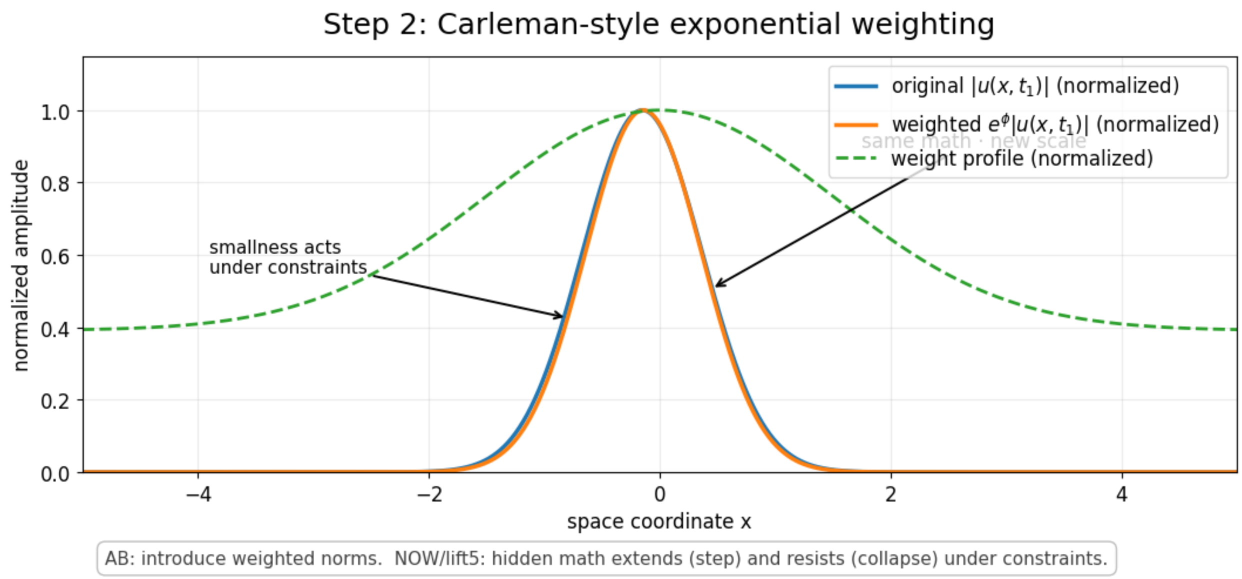

Figure 2: Step 2. Exponential (Carleman) weighting: same math, new scale.

6 CGCS translation layer

Standard step	Description	Function
two-time decay	paired input	test solution twice
Carleman weight	scale change	enforce constraints
Hardy inequality	variation cost	limit concentration
contradiction	mismatch	exclude non-zero
zero solution	remains	final result

A Reproducible notebooks and figures

This paper is accompanied by a set of computational notebooks that generate all figures. These notebooks are expository: they illustrate the structure of the proof pipeline but do not replace the formal estimates.

A.1 Notebook structure

- `01_decay_weight_visualization.ipynb` Illustrates two-time decay and Carleman-style exponential weighting.
- `02_hardy_gate_demo.ipynb` Shows how concentration incurs variation cost via Hardy-type constraints.
- `03_contradiction_pipeline_pde_aligned.ipynb` Combines weighted norms, variation cost, and curvature proxies to demonstrate the contradiction structure.

A.2 Figure provenance

Each figure in the paper is generated directly from the notebooks:

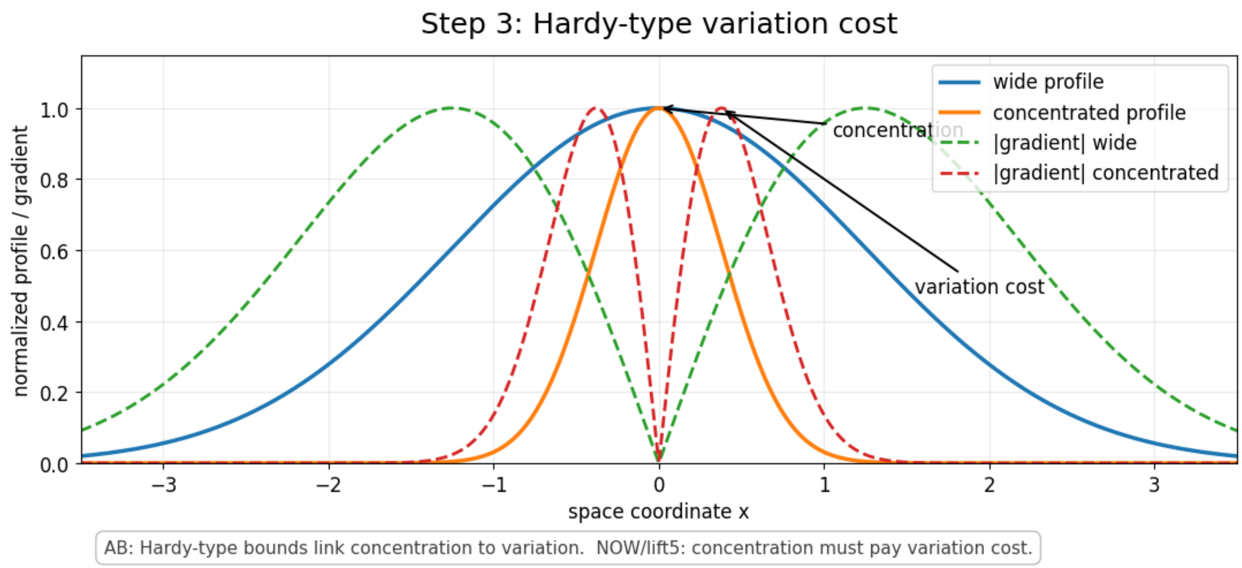

Figure 3: Step 3. Hardy variation cost: concentration must pay variation cost.

Figure	Source notebook
Step 1	Notebook 01
Step 2	Notebook 01
Step 3	Notebook 02
Step 4	Notebook 03
Step 5	Notebook 03

A.3 Interpretation vs proof

The quantities computed in the notebooks (weighted norms, gradient costs, curvature proxies) are finite-grid analogues of terms appearing in the formal analysis.

They are used to:

- visualize constraint interactions,
- illustrate incompatibility for non-zero candidates,
- support the logical structure of the contradiction argument.

They provide finite-grid realizations of the proof components in [1, 2], mapping weighted norms, variation costs, and curvature effects into directly computed quantities that reflect the structure of the analytical estimates.

A.4 Access

The notebooks and figures are available in the repository:

`https://github.com/thinkthoughts/unique-continuation-constraint-lab`

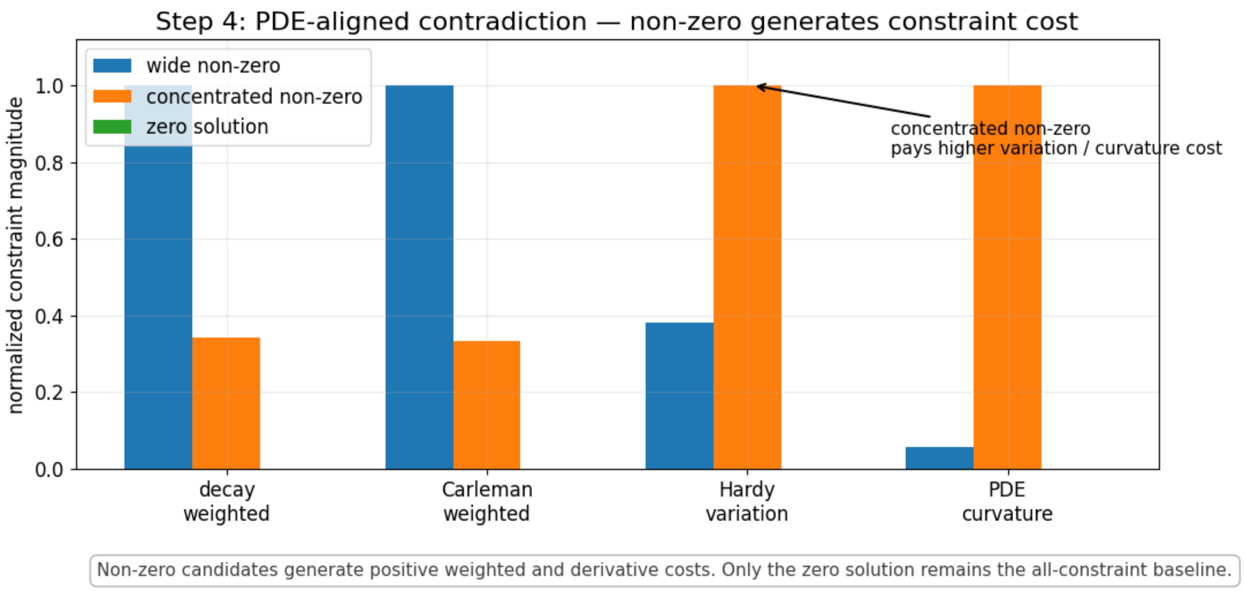

Figure 4: Step 4. PDE-aligned contradiction: non-zero candidates generate incompatible constraint costs.

A.5 Summary

The appendix connects:

$$\text{analysis} \leftrightarrow \text{computation} \leftrightarrow \text{visualization}.$$

This alignment is intended to improve clarity without altering the underlying mathematics.

References

[1] Luis Escauriaza, Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Hardy's uncertainty principle, convexity and schrödinger evolutions. *Journal of the European Mathematical Society*, 10(4):883–907, 2008.

[2] Terence Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*, volume 106 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, 2006.

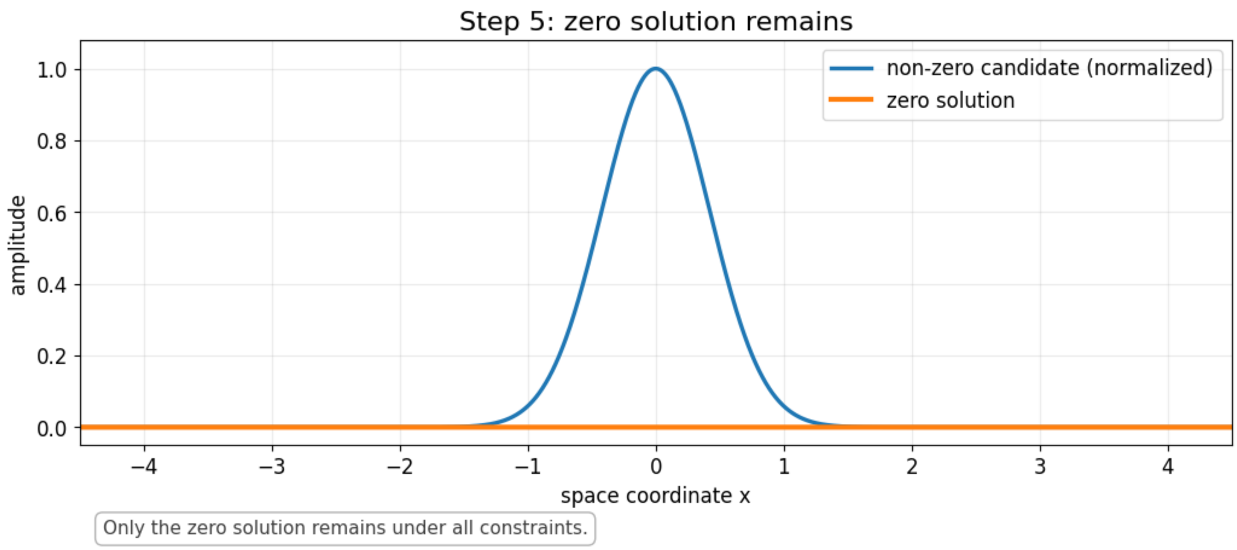

Figure 5: Step 5. Zero solution: only the zero solution satisfies all constraints.

Notebook 03: PDE-style constraint table

	decay weighted	Carleman weighted	Hardy variation	PDE curvature
wide non-zero	1.000	1.000	0.382	0.056
concentrated non-zero	0.343	0.334	1.000	1.000
zero solution	0.000	0.000	0.000	0.000

Figure 6: PDE-style constraint table: non-zero candidates generate positive costs, while the zero solution is the baseline.